Unit calculations, unit conversions, and all that

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A lot of people have trouble with calculations involving unit conversions. A typical problem might be, "a car travels 60 mph for 10 km. How long does that take in seconds? There are about 1.6 km in a mile." Do I divide? Multiply? Both? When? Where? Why? We will deal with most of these questions soon.

First, here's an easier problem. A car travels 60 mph for 3 hours. How far does the car go in miles? You probably know how to do this. Multiply $60 \times 3 = 180$ and you are done. But let's slow that down. Understanding simpler problems like this more deeply can make the harder ones easier.

The main insight is that *you can consider units (miles, hours, etc) to be just like numeric variables* that you multipy by the quantities. For example, "3 hours" can be treated like $3 \times h$ much like "3 fours" can be treated like 3×4. Another insight that goes along with that is that *the word "per" can be considered like the bar in a fraction*. So "mph" can be considered as mi/h or $\frac{\mathrm{mi}}{h}$.

Let's go back to our easier problem. We rewrite 60 mph as $60\,\frac{\mathrm{mi}}{h}$ and write 3 hours as $3h$. If we multiply those together, with the units left in, to get distance, we get

$$\begin{aligned}
60\,\frac{\mathrm{mi}}{h} \times 3h &= \frac{60}{1} \cdot \frac{\mathrm{mi}}{h} \cdot \frac{3}{1} \cdot \frac{h}{1} && \text{(multiplying fractions)} \\
&= \frac{180 \cdot \mathrm{mi} \cdot h}{h} && \text{(simplifying)} \\
&= \frac{180 \cdot \mathrm{mi}}{1} \cdot \frac{h}{h} \\
&= \frac{180 \cdot \mathrm{mi}}{1} \cdot 1 && \text{(simplifying)} \\
&= \frac{180\,\mathrm{mi}}{1} \\
&= 180\,\mathrm{mi}
\end{aligned}$$

We multiply the units just as if they are variables. But what good was all that when we already knew how to solve the problem before? The main advantage is that keeping track of the units is an important way to check our answers. Notice that we got 180 mi, not just 180, which gives us some confidence that we got it right. If we had decided to divide instead we might have gotten

$$\begin{aligned}
\frac{60\,\frac{\mathrm{mi}}{h}}{3h} &= 20\,\frac{\left(\frac{\mathrm{mi}}{h}\right)}{h} \\
&= 20 \cdot \frac{\mathrm{mi}}{h} \cdot \frac{1}{h} \\
&= 20\,\frac{\mathrm{mi}}{h^2}
\end{aligned}$$

Since the result is in strange $\frac{\mathrm{mi}}{h^2}$ units, not miles, we know our answer is wrong.

Now let's go back to the original problem: "a car travels 60 mph for 10 km. How long does that take in minutes? There are about 1.6 km in a mile." To do unit conversions, we add another insight: if we assume that $1.6\,\mathrm{km} = 1\,\mathrm{mi}$, then dividing one by the other should equal one:

$$\frac{1.6\,\mathrm{km}}{1\,\mathrm{mi}} = \frac{1\,\mathrm{mi}}{1.6\,\mathrm{km}} = 1$$

Similarly,

$$\frac{60\,\text{min}}{1\,h} = \frac{1\,h}{60\,\text{min}} = 1$$

Be sure to realize that these *unit conversions are nothing very special: they just involve multiplying by these alternate forms of one.*

So then 10 km can be converted to miles thus:

$$10 \cdot \text{km} \cdot \frac{1 \cdot \text{mi}}{1.6 \cdot \text{km}} = \frac{10 \cdot \text{mi} \cdot \text{km}}{1.6 \cdot \text{km}} = \frac{10 \cdot \text{mi}}{1.6} = 6.25 \cdot \text{mi}$$

Similarly, $60\,\frac{\text{mi}}{h}$ can be converted to $\frac{\text{mi}}{\text{min}}$ thus:

$$60\,\frac{\text{mi}}{h} \cdot \frac{1\,h}{60\,\text{min}} = \frac{60\,\text{mi}\,h}{60\,\text{min}\,h} = 1\frac{\text{mi}}{\text{min}}$$

Putting it all together,

$$\begin{aligned} \frac{6.25\,\text{mi}}{1\,\frac{\text{mi}}{\text{min}}} &= \frac{6.25\,\text{mi}\,\text{min}}{1\,\text{mi}} \\ &= 6.25\,\text{min} \end{aligned}$$

That's it!

If you like, you can do all the conversions in one big calculation thus:

$$\begin{aligned} \frac{10\,\text{km}\,\frac{1\,\text{mi}}{1.6\,\text{km}}}{60\,\frac{\text{mi}}{h}\,\frac{1\,h}{60\,\text{min}}} &= \frac{\left(\frac{10\,\text{km}\,\text{mi}}{1.6\,\text{km}}\right)}{\left(\frac{60\,\text{mi}\,h}{60\,\text{min}\,h}\right)} \\ &= \frac{6.25 \cdot \frac{\text{km}\,\text{mi}}{\text{km}}}{1 \cdot \frac{\text{mi}\,h}{\text{min}\,h}} \\ &= 6.25\,\frac{\text{mi}}{\left(\frac{\text{mi}}{\text{min}}\right)} \\ &= 6.25\,\text{mi} \cdot \frac{\text{min}}{\text{mi}} \\ &= 6.25\,\text{min} \cdot \frac{\text{mi}}{\text{mi}} \\ &= 6.25\,\text{min} \end{aligned}$$

Again, if you multiply when you should divide, or multiply or divide the wrong things, you will likely get units you are not expecting, which is evidence that you did something wrong.