Synthetic Division

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The parts of mathematics that work but that we don't understand can seem like magic. As we manage to understand them, they may seem a bit less like magic but even more beautiful.

For me, For years, I never got around to trying to understand synthetic division. Each year, the topic flew by fast enough that I didn't get around to trying to figure it out. Cursory searches through textbooks and the Web didn't help. I taught it mechanically; kids learned the technique, and all was "well." But each year, my mathematical conscience nagged me to get it done. Finally I did. So can you!

In its simplest form, synthetic division allows us to divide a polynomial p by a linear function of the form $x - a$ more easily than with long division. Here is the technique, presented mechanically. To divide $p = x^3 - 6\,x^2 + 11\,x - 6$ by $x - 3$ we write the additive inverse (the negative) of the constant term of the divisor next to the coefficients of the polynomial in a framework. (Unlike long division, we conventionally don't write the powers of x.) We write missing terms as zero.

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & & & \\ \hline & & & & \end{array}$$

Then we bring down the first coefficient under the framework:

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Then we multiply the number representing the divisor (in this case the 3) by the coefficient at the bottom and write that product under the next coefficient:

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & 3 & & \\ \hline & 1 & & & \end{array}$$

Add up that column:

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & 3 & & \\ \hline & 1 & -3 & & \end{array}$$

Repeat those steps until done:

$$\begin{array}{r|rrrr} 3 & 1 & -6 & 11 & -6 \\ & & 3 & -9 & 6 \\ \hline & 1 & -3 & 2 & 0 \end{array}$$

All the coefficients at the bottom except for the last one are the coefficients of the quotient polynomial; the last one is the constant term of the remainder:

$$(x^3 - 6\,x^2 + 11\,x - 6) \div (x - 3) = 1\,x^2 - 3\,x + 2\ (\text{remainder } 0)$$

or, equivalently, multiplying both sides by the divisor $x-3$,

$$x^3-6\,x^2+11\,x-6=(x-3)\,(1\,x^2-3\,x+2\,)+0 \tag{1}$$

So, why does this work?

To keep things simpler, let's focus on dividing where there's no remainder.

A key to understanding this process, like so many others, is to work backwards. To understand why $6\div 2=3$ it helps to realize that $6=2\cdot 3$. Similarly, we'll start with our divisor, $x-a$, and our quotient polynomial, q, and multiply them see what our dividend polynomial, p, will have to look like.

To get a little big concrete, let's define our quotient, q, as a generic polynomial of degree 3. (Once you work through this, you might try it with a polynomial of a different degree.)

$$q=b\,x^3+c\,x^2+d\,x+e$$

Now multiply that by $x-a$ to get our dividend, p (remember, $\frac{p}{x-a}=q$ so $p{=}(x-a)\,q$.

$$\begin{aligned} p &= (x-a)\,q \\ &= x\,q-a\,q \\ &= x\,(b\,x^3+c\,x^2+d\,x+e)-a\,(b\,x^3+c\,x^2+d\,x+e) \\ &= (b\,x^4+c\,x^3+d\,x^2+e\,x)-((a b)\,x^3+(a c)\,x^2+(a\,d)\,x+a\,e) \\ &= b\,x^4+(c-a\,b)x^3+(d-a\,c)\,x^2+(e-a\,d)\,x+(-a\,e) \end{aligned} \tag{2}$$

Now let's divide that product, $(x-a)\,q$, by $x-a$ using synthetic division and see that we should get q back. If you haven't gotten out your pencil yet (why not?) you definitely want to take it out now to work this out. The right hand column (with the equal signs) is not part of the normal synthetic division process but it's there so you can see how the parts fit together.

$$\begin{array}{r|ccccc} a & b & c-a\,b & d-a\,c & e-a\,d & -a\,e \\ & & a\,b & a\,c & a\,d & a\,e \\ \hline & b & c & d & e & 0 \end{array} \qquad \begin{array}{l} =(x-a)\,q \\ =a\,q \\ =x\,q \end{array} \tag{3}$$

Notice how the division, equation (3), looks just like the multiplication, equation (2), but upside-down (and with the powers of x missing). You can look at the bottom row of equation (3) as $x\,q$, the middle row as $a\,q$, and the top row as $p=(x-a)\,q$. (See the right hand column.) Since the coefficients of $x\,q$ and q are the same, we have found q at the bottom! (To keep this simple, I've ignored the remainder, which in this case is 0 because we chose p as the product $(x-a)\,q$.)

Is this magic? It's one thing to divide when we already know the quotient q, but how does this work when we don't? The trick is that we already know the coefficient, b, of the first term of q when we start, since must be the same as the coefficient of the first term of p. Now that we know that, notice that a times that first coefficient ($a\,b$ in the second row of equation (3)) is exactly what we need to add to the second coefficient of p (shown above as $c-a\,b$) to get c, the coefficient of the second term of p. Now that we know c, we can add $a\,c$ to the third term to get the third coefficient d, and so on down the line.

It's even more beautiful because it's *not* magic!