That Annoying SAT Circle Question

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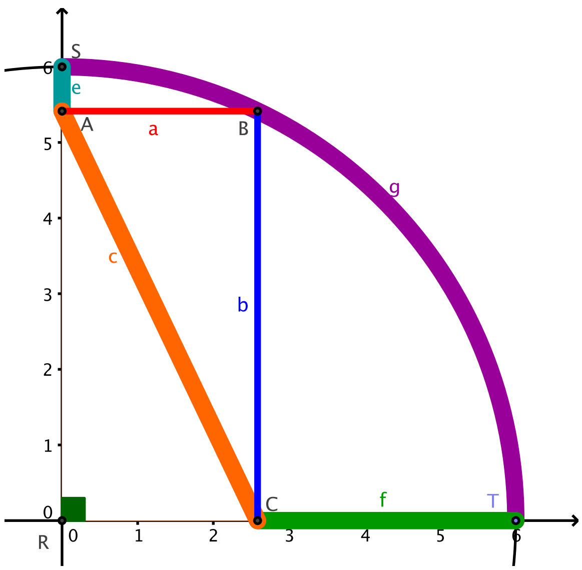
An SAT question that once stumped me (embarrassingly, since I was working with a student!) starts with something like this picture:



It asks a simple question:

Given that $a+b=8$ and that the radius of the circle is 6, what is the perimeter of the area bounded by e, c, f and g?

At this point, the best thing to do before proceeding is to try to solve the problem on your own. After you make a serious attempt, whether or not you succeed, read on. (This is true any time you are about to read an explanation of how to solve a math problem!)

This looks like a "divide and conquer" kind of problem. Just find e, c, f, and g. Then add them up. This approach seems to work for awhile. Let's start with g.

To find g, just notice that g is just $\frac{1}{4}$ $(\frac{90}{360})$ of a circle whose radius is 6, so

$$\begin{aligned} g &= \frac{1}{4}\cdot 2\pi\cdot r \\ &= \frac{1}{2}\pi\cdot 6 \\ &= 3\pi \end{aligned}$$

Finding c appears hard until you realize it is one of the two diagonals of rectangle $ABCR$. The *other* diagonal is a radius of the circle, and we already know that the radius is 6, so

$$c=6$$

But finding e and f is harder.

If we can't find e and f, it's worth asking whether you really need them. After all we are really trying to find $g+c+e+f$. Since we already know g and c, might it be easier to find $e+f$ rather than to find each individually? That turns out to be true.

An important problem-solving (and test-taking!) principle here is not to divide-and-conquer more than is necessary. Here's a completely different, much simpler problem that shows this principle:

> If $m+n=4$ and $p+q+r=6$, what is the average of m, n, p, q and r? Normally to find an average we would try to find all those values, then add, and then divide by 5 (the number of values being averaged). But in this case, we can't find the values. We notice instead that
>
> $$\begin{aligned} \text{average} &= \frac{m+n+p+q+r}{5} \\ &= \frac{(m+n)+(p+q+r)}{5} \\ &= \frac{4+6}{5} \\ &= 2 \end{aligned}$$
>
> Notice that we never had to find the individual values!

[In what follows, lower case letters represent segment lengths and upper case letters represent points.]

The Easy Way

It turns out that we can more easily find the sum $e+f$ than we can find either e or f individually. That's what I call the "easy way":

The radius is still 6 so

$$RS = RT = 6 \tag{1}$$

They also told us that

$$a+b=8 \tag{2}$$

From that and the fact that $ABCR$ is a rectangle,

$$AR+CR=8 \tag{3}$$

Since RS and RT are radii,

$$RS+RT=6+6=12 \tag{4}$$

Since the picture tells us that $AS+AR=RS$ and $CR+CT=RT$,

$$\begin{aligned} RS+RT &= 12 \\ AS+AR+CR+CT &= 12 \\ AR+CR+AS+CT &= 12 \text{ (commutative law)} \\ 8+AS+CT &= 12 \text{ (see eqn 3)} \\ AS+CT &= 4 \\ e+f &= 4 \end{aligned} \tag{5}$$

So the whole perimiter is

$$\begin{aligned} e+f+c+g &= e+f+6+3\pi \\ &= 4+6+3\pi \\ &= 10+3\pi \end{aligned} \tag{6}$$

The hard way

Just for fun, let's see what happens if we do it the "hard way" (find e and f separately). The Pythagorean Theorem says that

$$a^2+b^2=c^2=6^2=36$$

Substituting $8-b$ for a from equation (2):

$$(8-b)^2+b^2=36$$

Simplifying:

$$\begin{aligned}(8-b)^2+b^2 &= 36\\ 64-16b+b^2+b^2 &= 36\\ 64-16b+2b^2 &= 36\\ 2b^2-16b+28 &= 0\\ b^2-8b+14 &= 0\end{aligned}$$

We cannot factor the left side of that equation. So it's time to pull out the big gun for quadratic equations: the Quadratic Formula. I'm going to rewrite that equation using the variable x so we don't confuse the variable in the equation (now called x) with the b in the quadratic formula (coefficient of x term):

$$\begin{aligned}x^2-8x+14 &= 0\\ x &= \frac{8\pm\sqrt{64-4(1)(14)}}{2(1)}\\ &= \frac{8\pm\sqrt{64-56}}{2}\\ &= \frac{8\pm\sqrt{8}}{2}\\ &= 4\pm\sqrt{2}\end{aligned}$$

So the two possible values for b, (which we were just calling x), are

$$b=4\pm\sqrt{2} \tag{7}$$

To keep things simple, let's make an arbitrary decision:

$$b = 4+\sqrt{2}$$

Since $a+b=8$, then

$$\begin{aligned}a &= 8-b\\ &= 8-\left(4+\sqrt{2}\right)\\ &= 4-\sqrt{2}\end{aligned}$$

(If it were the other way around, it would work similarly.)

So then because e and b make up a radius, we have

$$e+b=6$$

Substituting, we get

$$\begin{aligned} e &= 6 - b \\ &= 6 - \left(4 + \sqrt{2}\right) \\ &= 2 - \sqrt{2} \end{aligned}$$

Similarly, because f and a also make up a radius,

$$\begin{aligned} f &= 6 - a \\ &= 6 - \left(4 - \sqrt{2}\right) \\ &= 2 + \sqrt{2} \end{aligned}$$

Do those e and f values give us the same perimeter as the "easy way"?

$$\begin{aligned} e + f + c + g &= \left(2 - \sqrt{2}\right) + \left(2 + \sqrt{2}\right) + 6 + 3\,\pi \\ &= 4 + 6 + 3\pi \\ &= 10 + 3\pi \end{aligned}$$

So the punch line is that if we solve this problem a very different, and harder, way (finding e and f by finding a and b using the Pythagorean Theorem and the Quadratic Formula) we get the same result as the "easy" way (in which we note that $e + f = 4$ and ignore the problem of finding e and f indivudually).

As one of my students said, "Math works! Who knew?"