Factoring Quadratics

by Jon Dreyer

Many people have relatively little trouble learning to factor (some) quadratics of the form x^2+bx+c, but throwing in an a (a standard-form quadratic of the form ax^2+bx+c) throws people for a loop. I have seen many presentations of how to factor this kind of quadratic, but this one also shows why the method works.

First let's examine how we factor quadratics like x^2+5x+6. Factoring means finding things that multiply to get the thing we are trying to factor. In the case of quadratics, that means finding two linear factors that multiply to get the original quadratic. Often a useful technique is to make a box in which the insides represent the quadratic, and to the left and on top we write the factors that multiply out to get what's in the box. This is a visual version of the distributive law. Here is a solution:

$\times$	x	$+2$
x	x^2	$2x$
$+3$	$3x$	6

This represents the equation

$$(x+3)(x+2)=x^2+3x+2x+6=x^2+5x+6$$

Note that the contents of each quadrant of the box is the product of the thing to its left and the thing above it. The "hard part" of factoring the quadratic was picking the 3 and the 2: the numbers that add up to 5 (b in the standard form quadratic) and multiply to 6 (c in the standard form quadratic). Notice that, since $a=1$ in this quadratic, $c=ac$ so we could also say that 3 and 2 multiply to ac. Let's focus on making them multiply to ac because that method works for harder quadratics also: cases when $a\neq 1$.

Many of us also focus on the 3 in $x+3$ and the 2 in $x+2$ as the numbers that add to b and multiply to ac. That works in this case, but not in cases when $a\neq 1$. Instead focus on the 3 and 2 *inside* the box. This seems like a silly distinction now but it won't in the next problem.

Specific example with $a\neq 1$ using the "box" method

Now let's factor $4x^2-8x-5$.

1. Notice that, in this quadratic, $ac=4\cdot(-5)=-20$. Choose m and n such that $mn=ac=-20$ and $m+n=b=-8$. This gives us $m=-10$ and $n=2$. Look ahead to the "But wait!" in step 3 to understand why we do this. Now fill in the *insides* of a "factoring box" using the m and n you just picked:

$4x^2$	$2x$
$(-10)x$	-5

2. Let's start filling out the two factors of ax^2 ($4x^2$ here) on the outside of the box. We don't know the coefficients yet so we'll write them like this:

$\times$	kx	
jx	$4x^2$	$2x$
	$(-10)x$	-5

   Obviously $jk=a=4$ (otherwise $jx\cdot kx\neq 4x^2$). To get a pretty factorization that has only integers, we want a j that divides both coefficients in its row of the box (a, which is 4 here, and n, which is 2 here), and a k that also divides both coefficients of its column (a again and m, which is -10 here). This gives us $j=2$ and $k=2$:

$$\begin{array}{c|c|c|}
\times & 2\,x & \\ \hline
2\,x & 4x^2 & 2\,x \\ \hline
 & (-10)\,x & -5 \\ \hline
\end{array}$$

3. Now it's easy to fill out the rest mechanically. We need to be able to multiply the bottom left number by 2 to get -10, giving -5, and we need to be able to multiply the top right number by 2 to get 2, giving $+1$.

$$\begin{array}{c|c|c|}
\times & 2\,x & +1 \\ \hline
2\,x & 4x^2 & 2\,x \\ \hline
-5 & (-10)\,x & -5 \\ \hline
\end{array}$$

Notice that $-5 = \frac{-10}{2}$ and $1 = \frac{2}{2}$. But wait! How did we get "lucky" on the -5 at the bottom right: If we picked the -5 based on the first column and the $+1$ based on the first row, how did we know that the -5 and the $+1$ would multiply to get the bottom right cell? Well,

$$\begin{aligned}
(-5)(1) &= \left(\frac{-10}{2}\right)\left(\frac{2}{2}\right) \\
&= \frac{-10 \cdot 2}{2 \cdot 2} \quad \text{(multiplying fractions)} \\
&= \frac{4(-5)}{4} \quad \text{(because that's how we picked } -10, 2, 2 \text{ and } 2\text{)} \\
&= -5
\end{aligned}$$

Thus the factorization is $(2\,x - 5)(2\,x + 1)$. You can check that this factorization works by multiplying it out to make sure it equals the original quadratic, $4x^2 - 8x - 5$.

General case: factoring using the "box" method

The problem is now to factor the *general* standard-form quadratic, $ax^2 + bx + c$. As you read this general solution, compare with the specific example above. The steps are numbered the same.

1. Choose m and n such that

$$mn = ac \text{ and } m + n = b$$

(Look ahead to the "But wait!" below to see why.) Now fill in the insides of a "factoring box" using the m and n you just picked:

$$\begin{array}{|c|c|} \hline
ax^2 & n\,x \\ \hline
m\,x & c \\ \hline
\end{array}$$

2. Pick j and k such that

$$jk = a$$

For a nice, integer factorization, make sure j divides n and k divides m. (This is not always possible.) This gives us a box like this:

$$\begin{array}{c|c|c|}
\times & k\,x & \\ \hline
j\,x & ax^2 & n\,x \\ \hline
 & m\,x & c \\ \hline
\end{array}$$

3. Now fill out the rest mechanically:

$\times$	kx	$+\frac{n}{j}$
jx	ax^2	nx
$+\frac{m}{k}$	mx	c

We chose j and k so $jk = a$, so the x^2 term of the product is ax^2. The choice of m and n so $m + n = b$ ensures that the x term of the product of the factors is bx. But wait! Why is the constant term c? Remember that we insisted that $mn = ac$, and, again, that $jk = a$. Thus

$$\frac{m}{k} \cdot \frac{n}{j} = \frac{mn}{jk} = \frac{ac}{a} = c$$

This is really the "big idea": By choosing m, n, j and k properly, the rest of the box has to work:

$$\left(jx + \frac{m}{k}\right)\left(kx + \frac{n}{j}\right) = ax^2 + bx + c$$

Once you see that $\frac{m}{k} \cdot \frac{n}{j} = c$, the rest is easy.

You don't really have to remember that the constant terms are $\frac{m}{k}$ and $\frac{n}{j}$ if you use the box; once you have chosen m, n, j and k properly, the box does the rest.

General case: Factoring algebraically

We don't really need the box to do this at all. We can do the same steps just using algebra. Again, the step numbers are the same as the other examples.

To factor $ax^2 + bx + c$:

1. Find m and n such that

$$mn = ac \text{ and } m + n = b$$

2. Find j and k such that

$$jk = a$$

Normally when factoring we also want **k to divide m and for j to divide n.** This gives us a nice, integer factorization.

3. This gives us a factorization of $\left(jx + \frac{m}{k}\right)\left(kx + \frac{n}{j}\right)$.

Here's why it works, no box needed:

$$\begin{aligned}
\left(jx + \frac{m}{k}\right)\left(kx + \frac{n}{j}\right) &= (jk)x^2 + \left(\frac{m}{k}\right)kx + j\left(\frac{n}{j}\right)x + \frac{mn}{jk} \\
&= ax^2 + mx + nx + \frac{mn}{jk} \\
&= ax^2 + (m+n)x + \frac{mn}{jk} \\
&= ax^2 + bx + \frac{mn}{jk} \\
&= ax^2 + bx + \frac{ac}{a} \\
&= ax^2 + bx + c
\end{aligned}$$

Voilà!

The careful reader will notice that this algebraic argument follows the same path as the "box" method, just without the box.