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Reconstructivism in Middle School

Mathematical abstraction in a nutshell

The process of doing mathematics involves two fundamentally different processes. The first is the process of *generalizing* from the concrete to the abstract.The second is the process of deducing facts about those abstractions. The power of mathematics is that these deductions must be no less valid that the abstractions from which they are deduced. For example, once we have the abstractions of similar triangles and basic algebra, we can deduce the Pythagorean Theorem, which in turn gives us lots of useful information about real-world triangles.

Viewed another way, we start with the real world and develop abstractions of some of it. Then the deductions we make about those abstractions allows us to learn more about our world.

Mathematicians strive to deduce as much as they can from the fewest and least uncontroversial postulates as they can, because they know that the deductions are hypothetical with respect to those postulates. There is also an aesthetic, in which, in the name of elegance, mathematicians tend to ignore in print the thought processes that led to the wise postulates and to the clever deductions, presenting the work as an austere, minimalist collection of postulates, definitions and proofs. One of the extreme examples of this aesthetic was the Bourbaki group, which endeavored to derive as much of modern mathematics (or "mathematic," as they chose to call it, to emphasize its unity) from axiomatic set theory (Borel 1998). Thus the typical *presentation* of mathematics does little to help the reader to experience the thought processes that the presenter went through on the way to learning the mathematics presented. As we will see, this aesthetic can cause problems with mathematics education.

Over millennia, wise choices of useful abstractions, clever deductions about those abstractions, and useful techniques based on those deductions together constitute the canon of mathematical knowledge.I hope it is an uncontroversial statement to suggest that the goals of mathematics education should be to help students to learn this canon in such a way that they can understand it, use it, and potentially contribute to it.

A very short history of abstraction in 20th century mathematics education

In the early 20th century, abstraction faded from mathematics education as it became dominated by progressivist educators like William Kilpatrick. "Limiting education primarily to utilitarian skills sharply limited academic content, and this helped to justify the slow pace of student centered, discovery learning, the centerpiece of progressivism" (Klein 2003 p. 178).

With the "New Math" of the 1950s and 1960s, the pendulum swung toward abstraction. The movement was led largely by mathematicians, many of whom in turn were influenced by the pure axiomatic approach exemplified by the Bourbaki. While some aspects of New Math survive today (e.g. calculus in high school (Klein p. 184)) it was largely considered a failure. There are many reasons for this. One reason might be that, even though an experienced mathematician can usually construct intuitive examples of, and mental models for, abstract concepts, this is a skill that takes years to develop, while the New Math often took that skill for granted. "[The logical system, in the way it is usually taught,] gives only the end-product of the mathematical discovery and fails to bring about in the learner those processes by which mathematical discoveries are made" (Skemp 1971, p. 13, in Hershkowitz 1990). Another reason for New Math's fall from grace might be that teachers were not themselves expert enough in abstract mathematics to teach it well.

The demise of New Math made the pendulum swung again. Educators took the upper hand and progressivism returned as an apparently new idea (Klein p. 185). Student-centered progressivism morphed into student-centered constructivism, in which *only* student-constructed knowledge is "truly integrated and understood." The NCTM came out with its heavily constructivist *1989 NCTM Curriculum and Evaluation Standards for School Mathematics* just as the Bush administration was promoting "standards-based education," and the NSF funded numerous curricula aligned with the constructivist NCTM standards (Klein p 195). Not only did many of these curricula not teach much abstraction, they encouraged students to invent their own techniques, relatively uncritically. Sadly, as modern educational constructivists will proudly proclaim, because students invent them, they will probably remember them, whether they are correct or not.

Many mathematicians and parents were concerned that children were not learning the canon nor even basic techniques. This led to the "math wars" started by groups such as "Mathematically Correct," (2008) although the "war" was not just about math, but also about the abuse of constructivist theory to limit content. In 1996, E.D. Hirsch, Jr. wrote *The Schools We Need and Why We Don't Have Them* which criticized this form of teaching as "sometimes inescure in its results--insecure...in the content of what is remembered. Students 'discover' all sorts of things, some of the irrelevant...and some of them wrong." (Hirsch 1996 p. 134). On the next page he adds, "The term 'constructivism' has become a kind of magical incantation used to defend discovery learning, which is no more sanctioned by psychological theory than any other form of constructed learning" (p. 135). The wars are not over.

War or dilemma?

Throughout this conflict, it looks like we have the mathematicians and their sympathizers on the side of theory and technique, and educators on the side of student-centered, hyper-constructivist learning, as if only one side were right. Kay Merseth often speaks of educational *dilemmas*, both horns of which represent valid positions, and therefore which must be "more commonly managed than resolved" (Merseth 1997). I suggest that this war would be more profitably managed as a dilemma than fought. Few doubt the value of the mathematical canon, nor does anybody who understands the basics of modern educational theory doubt that students must construct, somehow, their own knowledge. Sitting on either horn of this dilemma, as it seems most of the math warriors have done, is preposterous. Pouring the canon down students' throats cannot work, nor can dropping students in a mathematical playground and hoping for the best. We have to manage this dilemma.

Here is a musical analogy. Jazz music is one of the most personal, creative art forms there is. Most performances are improvised on the spot, as if from whole cloth. But the only reason these performances sound good and sound like jazz is that the musicians have absorbed the canon. Good jazz musicians spend a lot of time studying the theory and copying, over and over, the masters' improvisations. Only in this way can they expect to inherit enough of the tradition so that they can improvise freely, but still informed by tradition. The rules in mathematics are much stricter, so the same methods of practice must apply even more stringently: only by learning the theory developed by the masters, and by practicing mathematics as practiced by the masters, can one hope to benefit from their rich intellectual history.

On the other hand, no jazz musician wants to sound like a copy of any particular master. When studying theory, a good jazz musician tries not to simply commit the rules to memory, but rather to understand them deeply so as to better be able to use them and in some cases even to invent their own contributions. When practicing the masters' improvisations, they "drill" but do not "kill." Instead, they strive to play as if they were improvising. In this way, they gradually inherit some of the masters' thought processes and hence some of their improvisational abilities. Similarly, no math student wants to copy mindlessly a proof or repeatedly practice a technique as an end to itself, and every student should spend some time trying to invent some math. But when there is content to be taught, if the student cannot invent it, the student instead +reads a math book, or listens to a teacher, actively, with a pencil (Dyer-Bennet 1972) and attempts to reconstruct the contents mentally, rather than passively attempting to absorb the material. Ideally, if a student practices a technique, the goal is to integrate the technique into his repertoire as if he had invented it, not simply to be able to repeat it unconsciously. The closer we come to this goal, the more the student is able to "stand on the shoulders of giants," whether to see farther as Newton did, or simply to gain something from the giants' vantage point as most of us do.

There are no "jazz wars" analogous to the math wars. Every jazz musician realizes the value of copying the masters and mastering the theoretical canon on the one hand, and also the value of improvisation on the other. They manage the dilemma. Every moment spent copying a master is a moment spent not improvising, and every moment spent improvising is a moment spent not learning the canon. So the jazz musician does some of each and doesn't get all bent out of shape about it, because he knows that it is all necessary. Imagine math education with no math wars, in which mindful practice and mindful study of the canon were valued, but in which student invention, discovery and play were also valued. In this case there would be no war, only management of the dilemma when faced, at any one time, with choosing between valued options.

Reconstructivism

What of the objection that constructivism tells us that, for real learning to occur, students must construct their own knowledge? E.D Hirsch Jr. claims that constructivist educators did not fully understand the constructivism of psychologists and epistemologists: "It is not the case, as constructivists imply, that only such self-discovered knowledge will be reliably understood and remembered. This incorrect claim plays on an ambiguity between the technical and nontechnical uses of the term 'construct'" (Klein 2003 p. 192). A philosophical constructivist claims that our minds always construct our own knowledge, even if it is spoon-fed to us, so philosophical constructivism is a pretty useless concept for educators. Psychological constructivism seems to imply that our minds must have the experience of invention to learn, but that does not rule out some heavy scaffolding to ensure that students "invent" what we want them to. Groen and Kieran (1983) interpret Piaget: "Real comprehension...implies its reinvention by the pupil" (italics mine). Much as a false choice between beans and peas is much more likely to appeal to a small child than a demand to eat his vegetables, similarly leading a child to the point where an important concept is within sight (scaffolding) might provide enough of the sense of discovery that the concept holds, yet still ensure that the child learns the desired content. This is very different from the "eat whatever you want, because learning how to eat is more important than what you eat" approach of some modern educators.

This is just another example of the dilemma. At any point in a student's education at which he has not grasped some technique or concept, the teacher must choose between adding more scaffolding, which hopefully leads the student to the desired technique or concept at the expense of lessening (but not eliminating!) the sense of invention, or letting the student continue to invent at the expense of content. This choice is not easy, but I believe it is a mistake to go too far down the path of the student's own invention at the expense of correctness. Mathematics is built upon logic, and even a single contradiction can lead to any proposition at all. So one erroneous contradiction can, at least in principle, bring down the whole edifice. And besides, there is a canon to learn.

I will take the liberty of calling this way of thinking "reconstructivism" to emphasize both that it is consistent with constructivism, at least in the philosophical and psychological uses of the term, if not the constructivist educators' use, and also to emphasize that it is really often a good idea to give a student enough scaffolding to be correct, even at the expense of some of the experience of invention.

Middle school

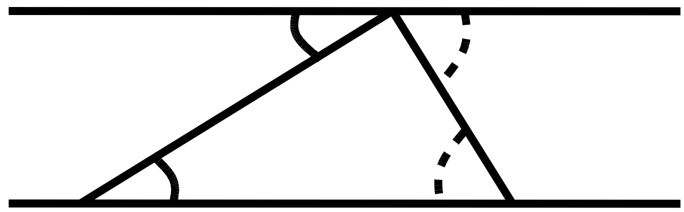
I bring up the concept of reconstructivism because I believe that it is particularly important at the middle school level to ensure that students learn mathematics correctly. Dahl (2004, p. 5) suggests that the unstable period of adolescence that is usually associated with *vulnerabilities* is also a unique period of *opportunities*. For an example, Dahl reminds us of the fact that it is much more difficult to learn fluency in a new language after puberty than during or before. In many ways, abstract mathematics is a language, and it might be that adolescence is the last opportunity for most of us to learn it fluently. So middle school may be the best time of life at which to learn abstract mathematical thinking. Yet Doda (2000 p. 45) laments that, in middle school at least, "misguided interpretations of of progressive instructional methods have yielded sloppy attention to intellectual development and authentic and substantive student learning."

Fortunately, middle school begins at roughly the time at which Piaget tells us the "propositional" or "formal operations" psychological stage begins (age 11-12) , and middle school ends roughly when this stage reaches equilibrium (age 14-15). This stage allows reasoning by hypothesis (Gruber & Vonèchein 1977 p. 461), the reasoning which is fundamental to abstract mathematical reasoning. So Piaget would suggest that it is possible to introduce abstract reasoning at this age. Reggiani (1994 p. 104) claims that even untrained students are capable of abstracting from the particular: "11-12 year old pupils not educated to think algebraically, nor trained to use formal properties, are able to 'sense the general" starting from particular cases." Rosskopf (1971 p. 128) adds, "By the time a child has finished the sixth grade he must have become familiar, at least in an intuitive way, with the properties of the number system: closure, commutativity, associativity, the identities for addition and multiplication, and the distributive property. Whether he can give the properties names or state them in terms of letters is not nearly so important as to have realized the properties--to have had experiences in which they appeared. Most eleven year olds are at the concrete operational stage of cognitive development. Some will be *at the threshold of the formal stage*. How they progress depends upon their own maturation patterns *and the sort of experiences they have in and out of school*" (italics added). Rosskopf also tells us that many low achievers never make it into the formal stage, leaving aside the question of whether, with the right teaching, these low achievers might otherwise reach it (p. 126). Indeed, Fischbein (1990) tells us that "every important progression in the child's reasoning capacity...may be achieved only as an effect of practice. The adolescent would never acquire spontaneously the formal perspective imposed by mathematics...the student learns mathematics...by constructing them through his or her own intellectual efforts. But individuals usually *do not do all these things by responding to their own problems and by resorting to their own, natural, intellectual means...The task of the teacher is to create an environment that would require a mathematical attitude, mathematical concepts, and mathematical solutions.*" In other words, we must heavily scaffold the student's construction of formal mathematics.

Even Glenda Lappan, co-author of the heavily constructivist *Connected Math* program says that middle school children "become capable of generalization, abstraction and argument in mathematics" (Lappan, p. 23). Unfortunately, while this leads her correctly to "expand their experiences with 'doing' mathematics" this does not lead her to recognize the value of "doing" it like the masters did.

It is clear that reconstructivist abstract mathematics will have to be taught differently from the New Math abstraction. Students will have to be guided in both directions, first in generalizing from the particular, and then in deducing from the general. An elegant proof may be a thing of beauty for someone schooled in that aesthetic, but that sort of appreciation takes practice. Balachef (1987 in Hershkowitz 1990 p.90) did research on a process by which 12-year-olds constructed a proof that the sum of the angles in a triangle is 180° by first doing experiments with triangles, then conjecturing the theorem, and finally proving the theorem "with the help of the teacher." A New Math approach, on the contrary, might start with the Euclidean axioms, progress to theorems about opposite interior angles and on to the proof, which would surely leave the students scratching their heads. In fact I tried an abbreviated version of this latter method with disastrous results. Another point to be made from this example is that good teaching involves lots of modeling, especially when on unfamiliar territory. Compare this with the modern educational constructivists, who would encourage middle school students to continue with their childish, fuzzy ways of thinking. Of course practicing this kind of fuzzy thinking reinforces it. It is interesting to me that "teacher talk" or "sage on the stage" teaching is commonly ridiculed today, yet if one uses the "educationally correct" synonym (or near-synonym, to be fair), "modeling," those same educators might nod in agreement.

Middle school abstraction does not necessarily have to be formal or explicitly axiomatic. Going back to the sum of the angles in a triangle, as long as we establish that it is "more obvious" that opposite interior angles are equal than that the sum of the angles is 180°, the following demonstration (suitably motivated and scaffolded) should be a big formal step above experimenting with paper triangles.



Before this demonstration, students should be encouraged to experiment with paper triangles as in Balachef's research. We could possibly even show a benefit of formalism over experimentation by giving some students bogus triangles (e.g. quadrilaterals with one nearly invisible vertex whose angle is almost, but not quite, 180°) which, if if fooled them, might help them appreciate exact formalisms as opposed to approximate experimentation. Rosskopf also supports the use of manipulatives at this stage.

Abstraction has practical power. A good abstraction is concise yet yields, via deduction, a wealth of consequences. For example, linear equations are a wonderful abstraction. In particular, linear equations can be used to find scale factors between similar geometric figures or ratios between their side lengths. Why, then, does the *Connected Math Stretching and Shrinking* booklet (2006) whose focus is on similar figures, ignore linear equations? Not only does this approach sacrifice an opportunity to use a powerful mathematical tool to find solutions, and not only does this approach sacrifice an opportunity to show the interconnectedness of mathematics, but it also gives students a lot of practice with inefficient and hard-to-generalize ways of thinking. Emerging formalists can also benefit from virtual worlds created by computer programs.

Seymour Papert in *Mindstorms* suggests that formalisms modeled by computer are much easier to learn because of the immediate, non-judgmental feedback (Papert 1980, p. 48) Hershkowitz also mentions a microworld in which formal rules are enforced by computer (Hershkowitz 1990 p. 91). In the case of the triangle sum, a geometry program could help along the way. Technology may also be useful specifically in helping students make the transition to algebraic thinking (Yerushalmy & Shierenberg 1994 p. 393).

Conclusion

I have chosen to focus on teaching middle school children because of the tremendous possibilities open at that unstable time of life. People climb the mountain of adolescence as children and come down as young adults, but the choices the adolescent makes on top, and the choices we help them to make, determine much of where the young adult will end up. I have chosen mathematics because it is a subject that I love and feel compelled to evangelize, and because it is one of the most important parts of any modern human being's education. I am also concerned by American students' poor performance in mathematics and feel that I know at least part of the reason for this performance: sloppy teaching that elevates, in the name of constructivism, the common students' inventions over the canon created by the effort over millennia of some of the finest minds ever to grace our planet. One constant throughout much of mathematical history has been a trend toward greater abstraction. A well-designed abstraction is simple yet wields tremendous power. But this kind of abstraction must be taught. Luckily it appears that most middle school students are capable of learning it. I believe that it is negligent of us as mathematics educators not to do so.

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